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THICK RECTANGULAR PLATES ON AN ELASTIC FOUNDATION

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THICK RECTANGULAR PLATES ON AN ELASTIC FOUNDATION

Daniel Frederick,* J.M. ASCE

ABSTRACT

The governing equations and solutions for the bending of thick rectangular plates resting on an elastic foundation are presented using the theory of plates developed by E. Reissner. A discussion of various types of fixity of edges involving three boundary conditions is presented and it is shown that Levy and Navier type solutions can be used. Also, two simple examples are included.

INTRODUCTION

The thick plate theory developed by E. Reissner[1, 2, 3] in 1945 is significant because it includes the effects of shear deformation and normal pressure. In addition, it permits the specification of three boundary conditions on an edge and requires the evaluation of just two functions. A theory analogous to that of Reissner's was developed by Mindlin[4] in 1951 for the motions of plates.

Since its development, Reissner's theory has been applied to the solution of several problems in the bending of plates. In 1952, Schafer[5] presented an example of the bending of a rectangular plate under a sinusoidal loading. Later, Naghdi and Rowley[6] extended the theory to include an elastic foundation which behaves according to the classical Winkler-Zimmerman assumption. A detailed history of the applications of the Reissner theory is contained in reference 7.

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7. Naghdi, P. M.: "The Effect of Elliptic Holes on the Bending of Thick Plates." Journal of Applied Mechanics, Paper No. 54-A-26.

This paper discusses the thick plate equations and their solutions for problems concerning the bending of rectangular plates resting on an elastic foundation. The basic equations are those obtained from the Reissner theory modified to include an elastic foundation in the manner of Naghdi and Rowley. A few simple numerical examples are also presented.

The Fundamental Equations

In order to conserve space, Reissner's notation will be used throughout this paper. For a detailed discussion of the notations and assumptions, the reader should refer to reference 3.

Fig. 1 shows the corrdinate system and the stress resultants M_x, M_y, M_{xy}, V_x, V_y in their positive directions. The displacements u, v and w are positive when they act toward the positive ends of the x, y and z axes respectively; and they are assumed to be of the form $u = z\alpha(x, y)$, $v = z\beta(x,y)$ and $w = \overline{w}(x,y)$. Here α and β are the rotations in the x and y directions respectively, of a line element originally perpendicular to the neutral surface and $\overline{w}$ is the vertical displacement of all points on this line element. They are defined as

$$\alpha = \frac{12}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}} uz\,dz, \quad \beta = \frac{12}{h^3}\int_{-\frac{h}{2}}^{\frac{h}{2}} vz\,dz, \quad \overline{w} = \frac{3}{2h}\int_{-\frac{h}{2}}^{\frac{h}{2}} w\left(1 - \frac{4z^2}{h^2}\right)dz \,. \tag{1}$$

Furthermore, it will be assumed that the stresses τ_{xz}, τ_{yz} and σ_z satisfy the equilibrium equations expressed in terms of the partial derivatives of the stress components and with no body forces and are therefore given by

$$\tau_{xz} = \frac{3V_x}{2h}\left(1 - \frac{4z^2}{h^2}\right), \quad \tau_{yz} = \frac{3V_y}{2h}\left(1 - \frac{4z^2}{h^2}\right),$$
$$\sigma_z = -\frac{q_1 + q_2}{2} + (q_2 - q_1)\left(\frac{3z}{2h} - \frac{2z^3}{h^3}\right). \tag{2}$$

In these equations q_1 is the pressure on the upper face and $q_2 = -kw$ (k is the foundation modulus) is the pressure on the lower face of the plate. The stress resultants are defined as in the classical theory and are quantities per unit of length.

Basic Equations Governing the Bending of a Rectangular Plate on an Elastic Foundation.

These equations arise from the application of Reissner's variational principle[3] taking into account the assumptions mentioned above. They also include shear stresses τ_{xzT} and τ_{yzT} on the top face of the plate and the shear stresses τ_{xyB} and τ_{yzB} on the bottom face of the plate. After some rearranging, these equations may be written as

$$D\nabla^4\bar{w} - \frac{h^2}{10}\frac{2-\nu}{1-\nu}\nabla^2(q_1 + k\bar{w}) + (q_1 + k\bar{w}) + \frac{h^3}{120}\frac{2-\nu}{1-\nu}\left(\frac{\partial}{\partial x}\nabla^2 S_x^+ + \frac{\partial}{\partial y}\nabla^2 S_y^+\right) - \frac{h}{2}\left(\frac{\partial S_x^+}{\partial x} + \frac{\partial S_y^+}{\partial y}\right) = 0.$$

$$\frac{h^2}{10}\nabla^2 V_x - V_x = D\frac{\partial}{\partial x}\nabla^2\bar{w} - \frac{h^2}{10(1-\nu)}\frac{\partial}{\partial x}(q_1 + k\bar{w}) + \frac{h^3}{120}\frac{2-\nu}{1-\nu}\frac{\partial^2 S_x^+}{\partial x^2} + \frac{h^3}{120}\frac{\partial^2 S_x^+}{\partial y^2} + \frac{h^3}{120(1-\nu)}\frac{\partial^2 S_y^+}{\partial x \partial y} - \frac{h}{2}S_x^+.$$

$$\frac{h^2}{10}\nabla^2 V_y - V_y = D\frac{\partial}{\partial y}\nabla^2\bar{w} - \frac{h^2}{10(1-\nu)}\frac{\partial}{\partial y}(q_1 + k\bar{w}) + \frac{h^3}{120}\frac{2-\nu}{1-\nu}\frac{\partial^2 S_y^+}{\partial y^2} + \frac{h^3}{120}\frac{\partial^2 S_y^+}{\partial x^2} + \frac{h^3}{120(1-\nu)}\frac{\partial^2 S_x^+}{\partial y \partial x} - \frac{h}{2}S_y^+. \qquad (3)$$

$$M_x = D\left\{\frac{6(1-\nu)}{5Gh}\frac{\partial V_x}{\partial x} - \left(\frac{\partial^2\bar{w}}{\partial x^2} + \nu\frac{\partial^2\bar{w}}{\partial y^2}\right) + \frac{3\nu}{5Gh}(q_1 + k\bar{w}) - \frac{2-\nu}{20G}\frac{\partial S_x^+}{\partial x} - \frac{\nu}{20G}\frac{\partial S_y^+}{\partial y}\right\}.$$

$$M_y = D\left\{\frac{6(1-\nu)}{5Gh}\frac{\partial V_y}{\partial y} - \left(\frac{\partial^2\bar{w}}{\partial y^2} + \nu\frac{\partial^2\bar{w}}{\partial x^2}\right) + \frac{3\nu}{5Gh}(q_1 + k\bar{w}) - \frac{2-\nu}{20G}\frac{\partial S_y^+}{\partial y} - \frac{\nu}{20G}\frac{\partial S_x^+}{\partial x}\right\}.$$

$$M_{xy} = \frac{Gh^3}{12}\left\{\frac{6}{5Gh}\left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x}\right) - 2\frac{\partial^2\bar{w}}{\partial x \partial y} - \frac{1}{10G}\left(\frac{\partial S_x^+}{\partial y} + \frac{\partial S_y^+}{\partial x}\right)\right\}$$

$$\alpha + \frac{\partial\bar{w}}{\partial x} = \frac{6}{5Gh}V_x \qquad (3)$$

$$\beta + \frac{\partial\bar{w}}{\partial y} = \frac{6}{5Gh}V_y$$

where

$$S_x^+ = \tau_{xzT} + \tau_{xzB} \qquad S_y^+ = \tau_{yzT} + \tau_{yzB}$$

$$S_x^- = \tau_{xzT} - \tau_{xzB} \qquad S_y^- = \tau_{yxT} - \tau_{yzB}.$$

The first six equations are the governing equations which contain six unknowns $\bar{w}$, V_x, V_y, M_x, M_y and M_{xy}.

Boundary Conditions.

In addition, the boundary conditions also arise from the variational principle and are three in number for each edge of the plate instead of two as in the classical theory. These were given by Reissner[1, 2, 3] and applications

involving a particular edge have been published by several authors. Reissner[2] has discussed a free edge, Schafer[5] describes one type of simply supported edge and Hirsch[8] gives boundary conditions at the edge of a rigid circular inclusion. Reference 7 summarizes the applications in which different boundary conditions are used.

Since three boundary conditions are to be satisfied, the usual definitions for the various types of fixity of an edge (simply supported, clamped, etc.) must be extended. These are discussed below. For the purpose of discussion, a straight edge perpendicular to the x axis will be used although the same statements apply to an arbitrary curvilinear boundary. The Reissner boundary conditions are (1) $M_x = M_x'$ or α prescribed, (2) $M_{xy} = M_{xy}'$ or β prescribed and (3) $V_x = V_x'$ or $\overline{w}$ prescribed. (Primed quantities are those prescribed on the boundary.)

(a) Simply Supported Edge.

In addition to the two usual conditions $\overline{w} = 0$ and $M_x = 0$, a third condition on either β or M_{xy} must be specified. This may be either $\beta = 0$ or $M_{xy} = 0$. Since M_{xy} is not zero on a simply supported edge in the classical theory, the classical definition is perhaps more closely approximated by choosing $\beta = 0$. Physically, this condition may be achieved by using a knife edge on the top and bottom of the plate if there is sufficient friction between the knife edges and the plate to prevent rotation of the line elements in the y direction. This scheme will prevent uplift at the corners which is annulled in the classical theory by using concentrated force on the corners. The other possibility, $M_{xy} = 0$, would require that there be no friction between the knife edges and the plate. This last set of conditions was used by Schafer[5] in his example of a rectangular plate under a sinusoidal loading.

(b) Free Edge.

The free edge was used by Reissner[2] in an application concerning an infinite plate with a circular hole free of stress. Here, the two conditions of the classical theory

$$M_x = 0 \qquad \text{and} \quad V_x + \frac{\partial M_{xy}}{\partial y} = 0$$

are replaced by $M_x = 0$, $M_{xy} = 0$ and $V_x = 0$. It is no longer necessary to transform the twisting moment into an equivalent vertical shear as in the classical theory. The vertical shear and the twisting moment each must vanish separately.

(c) Clamped Edge.

The usual conditions $\overline{w} = 0$ and $\frac{d\overline{w}}{dx} = 0$ will be replaced by (1) $\overline{w} = 0$, (2) $\alpha = 0$, or $\frac{d\overline{w}}{dx} = 0$ and (3) $\beta = 0$ or $M_{xy} = 0$. Although the clamped conditions may be never realized in an experiment, it appears that the condition $\alpha = 0$ is closer to fulfillment rather than the condition $\frac{d\overline{w}}{dx} = 0$. Also, since the statements made in the paragraph on the simply supported edge regarding the third condition, $\beta = 0$ or $M_{xy} = 0$ still apply, a clamped edge is perhaps best defined by $\overline{w} = 0$, $\alpha = 0$ and $\beta = 0$.

8. Hirsch. R. A. "The Effect of a Rigid Circular Inclusion on the Bending of a Thick Elastic Plate." Journal of Applied Mechanics, Vol. 19, 1952.

(d) Elastically Restrained Edge.

If a thick elastic plate frames into a beam, for which the governing equations are the usual classical ones which do not include the effect of shear deformation, then the boundary conditions for the plate can be written as:

$$(M_x)_{PLATE} = C \frac{\partial}{\partial y}\left(\frac{\partial^2 \bar{w}}{\partial x \partial y}\right)$$

$$(M_{xy})_{PLATE} = B \frac{\partial^3 w}{\partial y^3}$$

$$(V_x)_{PLATE} = B\left(\frac{\partial^4 \bar{w}}{\partial y^4}\right).$$

In these equations, B is the flexural and C is the torsional rigidity of the beam. These express the conditions that (a) the moment M_x in the plate must be associated with the twisting moment per unit length in the beam, (b) the twisting moment M_{xy} in the plate is equal to the bending moment per unit of length in the beam and (c) the vertical shear in the plate must be the load per unit of length on the beam. If the theory governing the bending of the beam includes shear deformation, then the beam terms would be modified accordingly.

Other types of edges may be reconsidered in the same manner.

A Method of Solving the Basic Equations.[9]

The basic equations (3) will now be considered for the case where $S_x^+ = S_y^+ = S_x^- = S_y^- = 0$. Solutions for this system of equations can be found from the equations on $\bar{w}$, V_x, and V_y. The general solution for $\bar{w}$ will be written as the sum of a particular integral and a complementary function as $\bar{w} = \bar{w}_H + \bar{w}_P$, where

$$\nabla^4 \bar{w}_H - \frac{c^2 k}{D} \nabla^2 \bar{w}_H + \frac{k}{D} \bar{w}_H = 0 \qquad (4)$$

and

$$\nabla^4 \bar{w}_P - \frac{c^2 k}{D} \nabla^2 \bar{w}_P + \frac{k}{D} \bar{w}_P = \frac{c^2}{D} \nabla^2 q_1 - \frac{q_1}{D} \qquad (5)$$

with the constant $c^2 = \frac{h^2}{10} \frac{2 - \nu}{1 - \nu}$.

Similarly, V_x and V_y can be written as the sum of a particular and complementary function as $V_x = V_{xH} + V_{xP}$ and $V_y = V_{yH} + V_{yP}$. If the function $\psi(x, y)$ is introduced, so that

$$V_{xH} = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad V_{yH} = -\frac{\partial \psi}{\partial x}, \qquad (6)$$

then the governing equation for ψ is

9. Frederick, D.: "On Some Problems in the Bending of Thick Plates." Ph.D. Dissertation, University of Michigan, 1955.

$$\frac{h^2}{10}\nabla^2\psi - \psi = 0. \tag{7}$$

The particular solutions for V_x and V_y can be written as

$$\begin{aligned} V_{xP} &= V_x^* + \frac{\partial}{\partial x}(-D\nabla^2\bar{w}_H + c^2 k\bar{w}_H) \\ V_{yP} &= V_y^* + \frac{\partial}{\partial y}(-D\nabla^2\bar{w}_H + c^2 k\bar{w}_H) \end{aligned} \tag{8}$$

where V_x^* and V_y^* are functions which must satisfy the following equations by virtue of the governing equations on V_x and V_y.

$$\begin{aligned} \frac{h^2}{10}\nabla^2 V_x^* - V_x^* &= D\frac{\partial}{\partial x}\nabla^2\bar{w}_P - \frac{h^2}{10(1-\nu)}\frac{\partial}{\partial x}(k\bar{w}_P + q_1) \\ \frac{h^2}{10}\nabla^2 V_y^* - V_y^* &= D\frac{\partial}{\partial y}\nabla^2\bar{w}_P - \frac{h^2}{10(1-\nu)}\frac{\partial}{\partial y}(k\bar{w}_P + q_1). \end{aligned} \tag{9}$$

Hence, for the rectangular plate, the complementary functions can be found from equations (4) and (7) and the constants of integration associated with these can be evaluated from the 12 boundary conditions which arise from the four edges of the plate.

Applications

In this section, two simple examples are presented and the results are compared numerically with the predictions of the classical theory. In addition, some general solutions are discussed.

Navier Type Solution.[10]

A rectangular plate of dimensions a by b has coordinate axes along two edges as shown in Fig. 2. It is simply supported on each of the four edges in the following sense:

$$\begin{aligned} M_x(o,y) = M_x(a,y) = M_y(x,o) &= M_y(x,b) = 0 \\ \bar{w}(o,y) = \bar{w}(a,y) = \overline{w}(x,o) &= \bar{w}(x,b) = 0 \\ \beta(o,y) = \beta(a,y) = \alpha(x,o) &= \alpha(x,b) = 0. \end{aligned} \tag{10}$$

Furthermore, it is assumed that the top and bottom faces are free of shearing stress and that the loading function $q_1(x,y)$ can be represented by a double Fourier Series.

$$q_1(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}. \tag{11}$$

The Fourier coefficients a_{mn} are determined in the usual way

10. Timoshenko, S.: "Theory of Plates and Shells," First Edition, McGraw-Hill Book Company, 1940.

$$a_{mn} = \frac{4}{ab} \int_0^a \int_0^b q_1(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dxdy. \tag{12}$$

With the assumptions above $\bar{w}$, V_x and V_y can be expressed in the Navier form and the double Fourier coefficients can be found in the usual way. The final expressions for $\bar{w}$, V_x and V_y are presented below.

where
$$\bar{w} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{13}$$

$$w_{mn} = -\frac{\left[\frac{c^2}{D}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} + \frac{1}{D}\right] a_{mn}}{\left\{\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right\} + \frac{k}{D}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} + \frac{k}{D}},$$

$$V_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$
$$V_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \tag{14}$$

with
$$b_{mn} = \frac{\left(\frac{m\pi}{a}\right)\left[D\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} w_{mn} + \frac{h^2}{10(1-\nu)}\left\{k w_{mn} + a_{mn}\right\}\right]}{\frac{h^2}{10}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} + 1}$$

$$c_{mn} = \frac{\left(\frac{n\pi}{b}\right)\left[D\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} w_{mn} + \frac{h^2}{10(1-\nu)}\left\{k w_{mn} + a_{mn}\right\}\right]}{\frac{h^2}{10}\left\{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right\} + 1}.$$

The solutions given by equations (13) and (14) satisfy the boundary conditions (10) and hence the problem is solved. The classical Navier solution satisfies the classical plate equations which are obtained by dropping the terms containing h^2 in the governing equations (3). It also satisfies the classical simple support boundary conditions which are the conditions (10) with the conditions on α and β omitted.

Levy Type Solution.[10]

The plate of the previous paragraph will be considered again with a different set of boundary conditions. It will be simply supported on two opposite edges in the same manner as above.

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10. See p. 6 ftn.

$$\begin{aligned} M_x(o,y) &= M_x(a,y) = 0 \\ \bar{w}(o,y) &= \bar{w}(a,y) = 0 \\ \beta(o,y) &= \beta(a,y) = 0. \end{aligned} \tag{15}$$

Furthermore, it may have any combination of boundary conditions on the other two edges consistent with the Reissner theory. In addition, it will be assumed that the function q_1 is independent of y and can be represented by the Fourier series

$$q_1(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{a}. \tag{16}$$

Again, the Fourier coefficients can be evaluated in the usual way and are

$$a_n = \frac{2}{a} \int_o^a q_1(x) \sin \frac{n\pi x}{a} \, dx. \tag{17}$$

Due to the even derivatives occuring in equation (5), it is obvious that the particular solution $\bar{w}_p$ can be written in the following form:

$$\bar{w}_P = \sum_{n=1}^{\infty} w_n \sin \frac{n\pi x}{a}. \tag{18}$$

Upon substituting equation (18) into equation (5) the coefficients w_n are found to be

$$w_n = - \frac{\left\{ 1 + \left(\frac{n\pi}{a}\right)^2 c^2 \right\} a_n}{D\left(\frac{n\pi}{a}\right)^4 + c^2 k \left(\frac{n\pi}{a}\right)^2 + k}.$$

With $\bar{w}_p$, V_x^* can be determined from the first of equations (9). This can be expressed in the same form since the left side of the governing equations contains the function and even derivatives of the function.

$$V_x^* = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{a} \tag{19}$$

where

$$c_n = \frac{Db_n \left(\frac{n\pi}{a}\right)^3 + \frac{h^2}{10(1-\nu)} (kb_n + a_n)\left(\frac{n\pi}{a}\right)}{1 + \frac{h^2}{10} \left(\frac{n\pi}{a}\right)^2}.$$

From the second of equations (9), it can be seen that $V_y^* = 0$, since the right side contains derivatives with respect to y while $\bar{w}_p$ and q_1 are independent of y.

With the boundary conditions (15) in mind it is evident that for this problem $\bar{w}_H$ can be taken in a Levy form.

$$\bar{w}_H = \sum_{n=1}^{\infty} Y_n(y) \sin \frac{n\pi x}{a} . \tag{20}$$

Upon substituting this into equation (4) and solving the resulting differential equation on $Y_n(y)$, $\bar{w}_H$ becomes

$$\bar{w}_H = \sum_{n=1}^{\infty} (A_n \cosh e_n y \cos f_n y + B_n \cosh e_n y \sin f_n y + C_n \sinh e_n y \cos f_n y + D_n \sinh e_n y \sin f_n y) \sin \frac{n\pi x}{a} \tag{21}$$

where A_n, B_n, C_n, and D_n are independent constants of integration and where

$$e_n = r_n^{\frac{1}{2}} \cos \frac{\theta_n}{2} , \qquad f_n = r_n^{\frac{1}{2}} \sin \frac{\theta_n}{2}$$

with

$$r_n = \left\{ \left(\frac{n\pi}{a}\right)^4 + \frac{kc^2}{D}\left(\frac{n\pi}{a}\right)^2 + \frac{k}{D} \right\}^{1/2} , \quad \theta_n = \tan^{-1} \frac{\left\{ \frac{k}{D} - \frac{kc^2}{4D^2} \right\}^{1/2}}{\left(\frac{n\pi}{a}\right)^2 + \frac{kc^2}{2D}} .$$

Again, in view of the boundary conditions (15), Ψ will be taken as

$$\psi(x,y) = \sum_{n=1}^{\infty} \bar{Y}_n(y) \sin \frac{n\pi x}{a} . \tag{22}$$

After substituting this into equation (7) and solving the resulting differential equation on $\bar{Y}_n(y)$, we find that the final form for Ψ can be written as

$$\psi(x,y) = \sum_{n=1}^{\infty} \left\{ G_n \cosh \beta_n y + H_n \sinh \beta_n y \right\} \sin \frac{n\pi x}{a} \tag{23}$$

where $\beta_n^2 = \alpha_n^2 + \frac{10}{h^2}$ and G_n and H_n are independent constants of integration.

With $\bar{w}_H$ and Ψ given by equations (21) and (23) respectively, the boundary conditions (15) are automatically satisfied. There still remains six independent constants which will be determined by the six boundary conditions on the two remaining edges of the plate. In the classical theory at the corresponding stage, the Levy solution has four constants which are determined by four boundary conditions on the two remaining edges of the plate.

The method described above also works for the case where two opposite edges of the plate have the boundary conditions

$$\alpha(o,y) = \alpha(a,y) = 0$$
$$V_x(o,y) = V_x(a,y) = 0 \qquad (24)$$
$$M_{xy}(o,y) = M_{xy}(a,y) = 0 ,$$

while the remaining edges have any combination of conditions. In this case, the solutions for $\overline{w}_H$ and Ψ are taken as products of a function of y and $\cos \frac{n\pi x}{a}$. The boundary conditions (24) apply to an infinite plate supported by equidistant columns and this problem also has a counterpart in the classical theory.

A Line on an Infinite Plate Resting on an Elastic Foundation.[9] (See p. 5 ftn.)

A line load of intensity P (force per unit length) acts along the y axis as shown in Fig. 3. Since the variable y does not enter into the solution, the governing equations (3) become

$$\frac{d^4\overline{w}}{dx^4} - \frac{kc^2}{D}\frac{d^2\overline{w}}{dx^2} + \frac{k}{D}\overline{w} = \frac{c^2}{D}\frac{d^2q_1}{dx^2} - \frac{q_1}{D}$$

$$\frac{h^2}{10}\frac{d^2V_x}{dx^2} - V_x = -D\frac{d^3\overline{w}}{dx^3} + \frac{h^2}{10(1-\nu)}\frac{d}{dx}(k\overline{w} + q_1)$$

$$\frac{h^2}{10}\frac{d^2V_y}{dx^2} - V_y = 0$$

$$M_x = \frac{h^2}{5}\frac{dV_x}{dx} - D\frac{d^2\overline{w}}{dx^2} + \frac{h^2\nu}{10(1-\nu)}(k\overline{w} + q_1)$$

$$M_y = -\nu D\frac{d^2\overline{w}}{dx^2} + \frac{h^2\nu}{10(1-\nu)}(k\overline{w} + q_1) \qquad (25)$$

$$M_{xy} = \frac{h^2}{10}\frac{\partial V_y}{\partial x}$$

$$\alpha + \frac{d\overline{w}}{dx} = \frac{6}{5Gh}V_x$$

$$\beta = \frac{6}{5Gh}V_y .$$

Due to the load, there will be a uniform deflection along the y axis. This will be denoted by w_o. Hence, the boundary conditions at the origin will be written as

$$\overline{w}(o) = -w_o$$
$$\alpha(o) = 0 \qquad (26)$$
$$\beta(o) = 0 .$$

The general method of solution described above in which a function Ψ was introduced will be used. For this particular example since Ψ is independent of y, the equation on Ψ becomes

$$\frac{h^2}{10}\frac{d^2\psi}{dx^2} - \psi = 0. \tag{27}$$

Using the condition that $\bar{w}$, M_x and V_x vanish as x approaches infinity, the solutions for ψ and $\bar{w}$ which are to be used take the following form

$$w = Ae^{-e_o x}\cos f_o x + Be^{-e_o x}\sin f_o x$$
$$\psi = Ge^{-\sqrt{\frac{10}{h}}x} \tag{28}$$

where

$$e_o = r^{\frac{1}{2}}\cos\frac{\theta}{2}, \qquad f_o = r^{\frac{1}{2}}\sin\frac{\theta}{2}$$

$$r = \left(\frac{k}{D}\right)^{\frac{1}{2}}, \qquad \theta = \tan^{-1}\frac{\left\{\frac{h}{D} - \frac{k^2c^4}{4D^2}\right\}^{1/2}}{\frac{kc^2}{2D}}.$$

Upon application of the boundary conditions (26), the constants A, B and G become

$$G = 0 \qquad A = w_o$$

$$B = -w_o\frac{e_o + \frac{6}{5Gh}\left\{De_o^3 - 3De_of_o^2 - kc^2e_o\right\}}{f_o - \frac{6}{5Gh}\left\{Df_o^3 - 3De_o^2f_o + kc^2f_o\right\}}. \tag{29}$$

With these constants, the expressions for M_x, V_x and M_y can be found from equations (25). It can also be seen that M_{xy} and V_y are zero. The above solution will now be compared with the classical solution where the boundary conditions are

$$\bar{w}(o) = -w_o \qquad \left(\frac{d\bar{w}}{dx}\right)_{x=0} = 0;$$

and the governing equations are those obtained from equations (25) by dropping the terms containing h^2. For convenience in the calculations, it has been assumed that $\frac{kh^4}{D} = 1$ for both theories. The deflections, moments and shears are shown in Fig. 4. In these and the figures which follow C denotes the results of the classical theory and R denotes the results of the Reissner theory.

As an alternative, the results may be expressed in terms of P instead of w_o through the relation $V_x(o) = \frac{P}{2}$. Fig. 5 shows the deflection, moment, shear and the stress σ_z in terms of the load P.

Since the bending stresses and shearing stresses in the plate are given by

the usual elementary strength of materials formulas, it can be seen from Figs. 4b and 4c that the maximum bending stress is reduced about 55% and the maximum shearing stress is lowered by about 15% over the corresponding values given by the classical theory. Thus, a design of the slab based upon the classical theory is "safe" but overdesigned.

From Figs. 5b and 5d it is seen that the same conclusions are valid for the case of a given line load pressing on the Reissner plate and a classical plate. The moments and shears are reduced by using a design based upon the Reissner theory. On the other hand, a glance at Fig. 5a shows that the maximum deflection by the Reissner theory is greater than that given by the classical theory. These trends appear to be true in general. In the particular case of the maximum deflection, it seems intuitively obvious that a theory including shear deformation should give a deflection larger than that given by a theory which neglects shear deformation.

Fig. 5d shows that the normal stress σ_z decreases as the distance from the load increases.

Bending of a Plate Due to a Uniform Shear Stress on the Top and Bottom Faces with Two Opposite Edges Clamped and of Infinite Length Parallel to These Edges.

To illustrate a boundary value problem involving shear stresses on the top and bottom faces, the plate shown in Fig. 6 will be considered. In order to simplify the problem as much as possible, the plate has been taken of infinite length in the y direction and the shear stress on the top and bottom faces has been taken as a constant. In addition, the foundation has been omitted since it would change only the form of the solution of $\overline{w}$. In practice this type of bending might occur in a bin full of material in which there is friction at the wall of the bin due to packing of the material. This friction would produce both bending and extension of the plate; however, this example illustrates a method for handling the bending only.

Since all quantities involved are independent of y, the governing equations as obtained from equations (3) are

$$S_y^+ = S_y^- = S_x^- = 0, \qquad S_x^+ = 2\tau$$

$$\frac{d^4\overline{w}}{dx^4} = 0$$

$$\frac{h^2}{10}\frac{d^2V_x}{dx^2} - V_x = D\frac{d^3\overline{w}}{dx^3} - h\tau \tag{30}$$

$$\frac{h^2}{10}\frac{d^2V_y}{dx^2} - V_y = 0$$

$$M_x = D\left\{\frac{6(1-\nu)}{5Gh}\frac{dV_x}{dx} - \frac{d^2\overline{w}}{dx^2}\right\}$$

$$M_y = -\nu D\frac{d^2\overline{w}}{dx^2}$$

$$M_{xy} = \frac{h^2}{10}\frac{dV_y}{dx} \tag{30}$$

$$\alpha + \frac{d\bar{w}}{dx} = \frac{6}{5Gh} V_x, \qquad \beta = \frac{6}{5Gh} V_y.$$

For the edges which are clamped, the boundary conditions will be taken as

$$\begin{aligned} \bar{w}(o) &= \bar{w}(L) = 0 \\ \alpha(o) &= \alpha(L) = 0 \\ \beta(o) &= \beta(L) = 0 \end{aligned} \tag{31}$$

Following a procedure similar to that used in the previous example, the solutions for $\bar{w}$ and ψ can be written as

$$\begin{aligned} \bar{w} &= a_o + a_1 x + a_2 x^2 + a_3 x^3 \\ \psi &= a_4 \sinh\sqrt{\frac{10}{h}}\, x + a_5 \cosh\sqrt{\frac{10}{h}}\, x . \end{aligned} \tag{32}$$

There are now six constants a_0, a_1, a_2, a_3, a_4, a_5 which are to be determined by the six boundary conditions (31). Omitting the detailed work, the final values of the costants are

$$a_o = a_4 = a_5 = 0$$

$$a_1 = \frac{3}{5}\frac{\tau L}{Gd}, \qquad a_2 = -\frac{9}{5}\frac{\tau}{Gd}, \qquad a_3 = \frac{6}{5}\frac{\tau}{GLd} \tag{33}$$

where

$$d = \frac{L}{2}\left\{1 + \frac{12h^2}{5(1-\nu)L^2}\right\}.$$

Hence, the expressions for $\bar{w}$, M_x and V_x become

$$w = \frac{3}{5}\frac{\tau L^2}{Gd}\left\{\left(\frac{x}{L}\right) - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3\right\}$$

$$V_x = \frac{\tau h L}{2d}, \qquad M_x = \frac{9}{10}\frac{\tau h^3}{d}\left\{1 - 2\left(\frac{x}{L}\right)\right\}. \tag{34}$$

A plot of these equations for $\nu = \frac{1}{3}$ is shown in Fig. 7 for different $\frac{h}{L}$ values. This problem can not be solved by the classical theory since the solution $\bar{w}$ would be the same as above, and the boundary conditions $\bar{w}(o) = \bar{w}(L) = \left(\frac{d\bar{w}}{dx}\right)_{x=0} = \left(\frac{d\bar{w}}{dx}\right)_{x=L} = 0$ would require that all four constants of integration be zero. However, by letting $\frac{h}{L} \to 0$ with $\lim_{h\to 0}(\tau h)$ remaining finite, one can get the solution for a plate of zero thickness subjected to a moment of magnitude τh at every point of the plate from the Reissner theory. These results are also drawn in Fig. 7. The bending moment M_x for $h = 0$ is zero everywhere.

CONCLUSION

The thick plate theory which was developed by E. Reissner, which includes the effects of shear deformation and normal pressure, can be used to solve certain problems in the bending of rectangular plates resting on an elastic foundation. In this paper two simple examples are presented and some results are plotted graphically. From the graphical plots, the thickness-length parameter at which the effects of shear deformation and normal pressure become important can be determined.

ACKNOWLEDGMENTS

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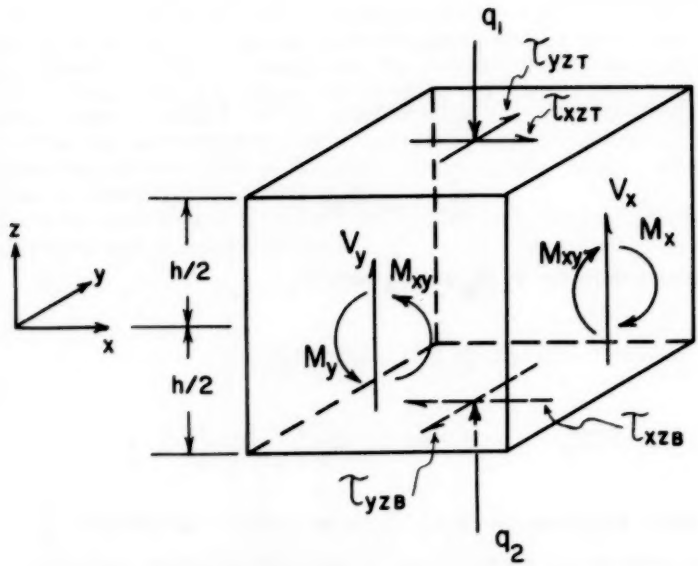


Figure 1 - An Element of the Plate for Rectangular Coordinates Showing the Stress Resultants.

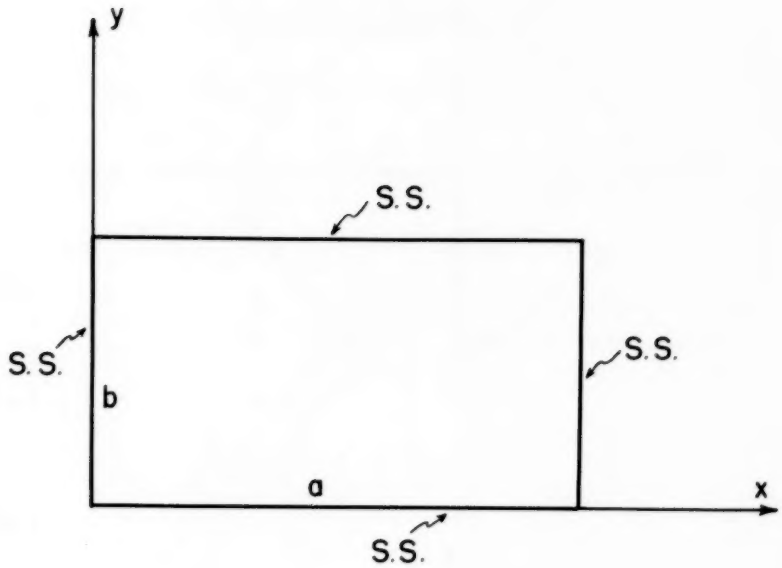


Figure 2 - A Rectangular Plate with Coordinate Axes.

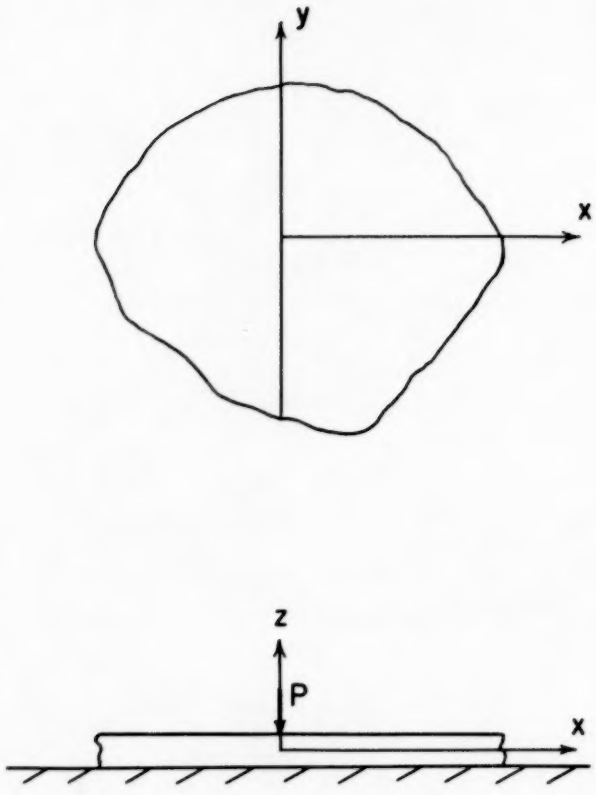
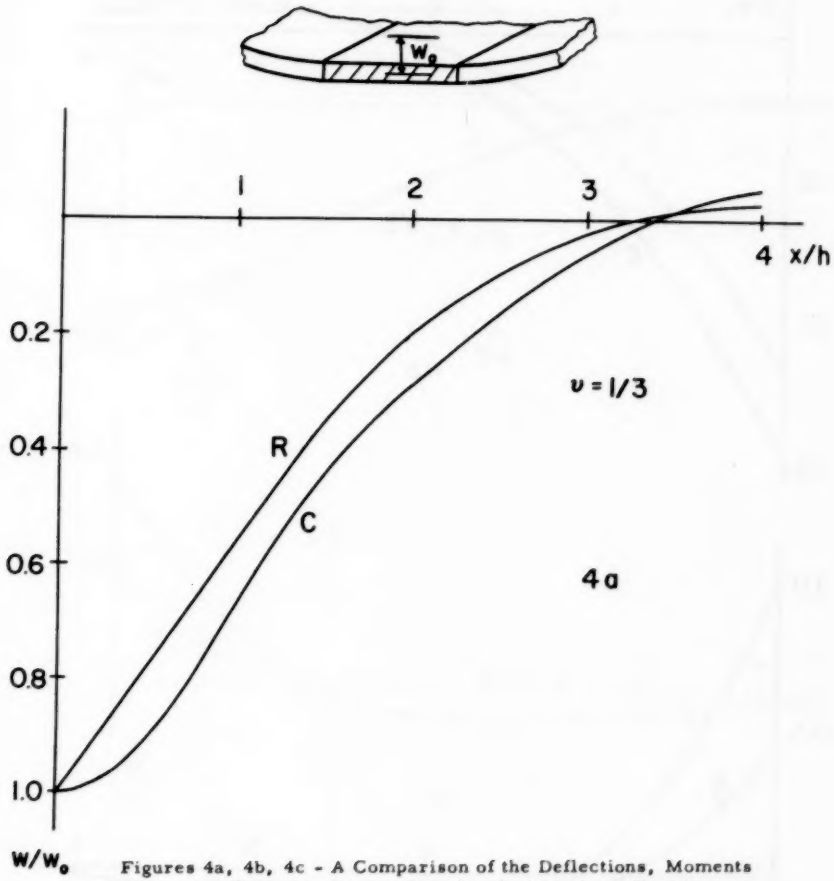
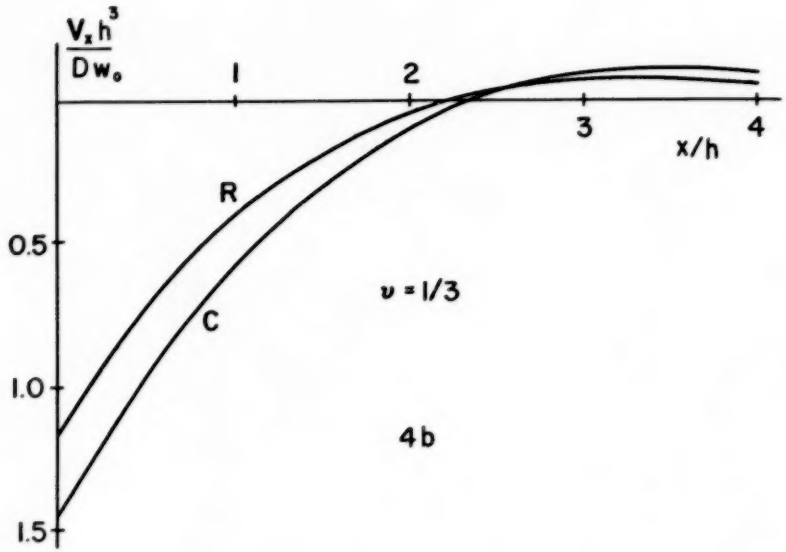


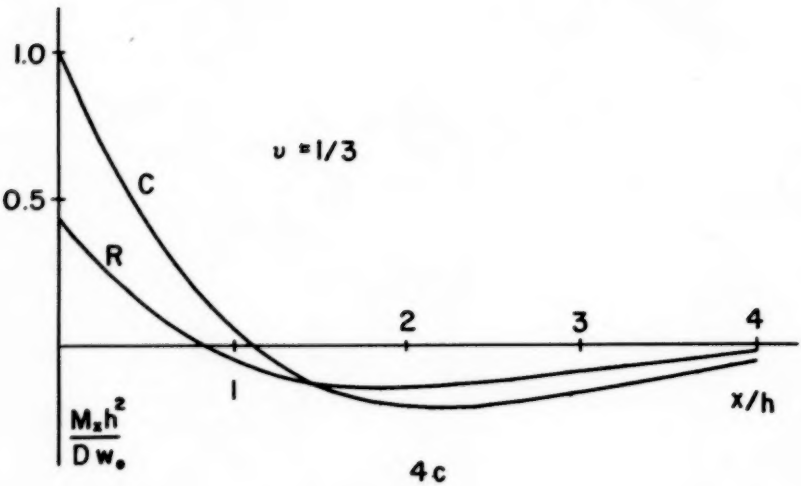
Figure 3 - An Infinite Plate with a Line Load of Intensity P.



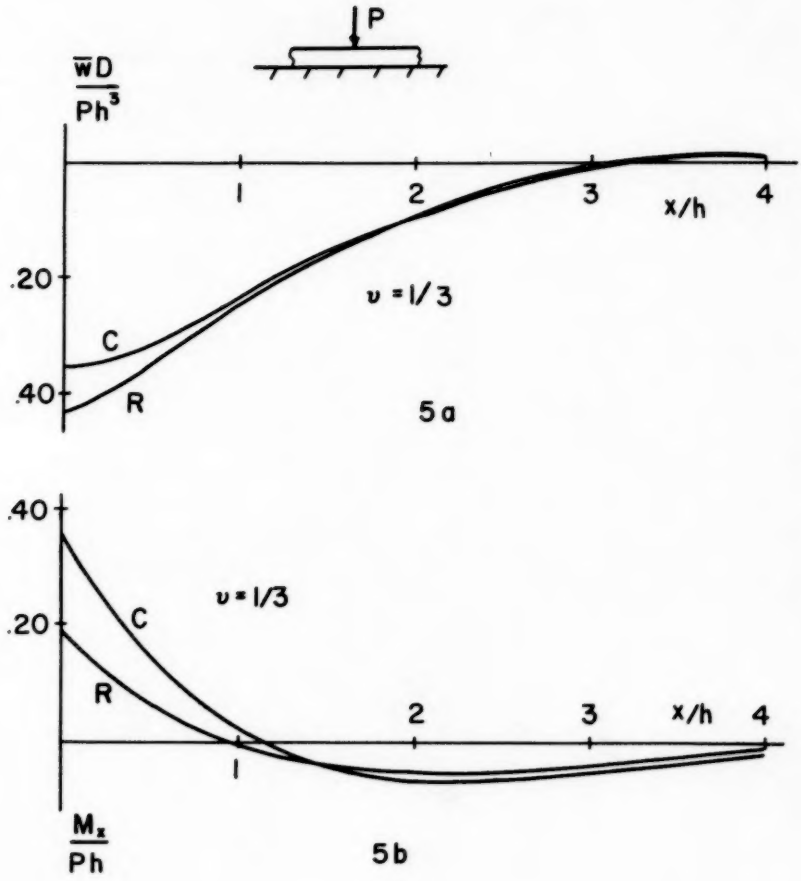
Figures 4a, 4b, 4c - A Comparison of the Deflections, Moments and Shears in terms of w_o for the Line Load Problem.


$\frac{V_x h^3}{D w_o}$
1
2
3
4
x/h
R
C
0.5
1.0
1.5
υ = 1/3

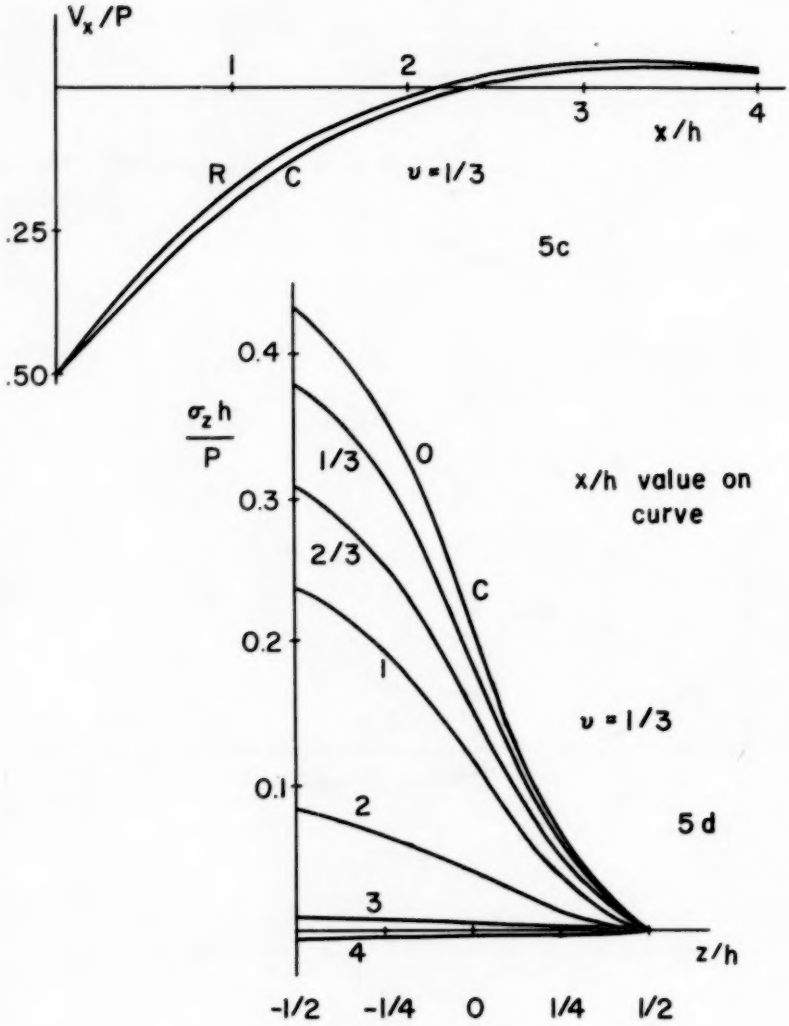
4b


1.0
0.5
υ = 1/3
C
R
2
3
4
1
x/h
$\frac{M_x h^2}{D w_o}$

4c



Figures 5a, 5b, 5c, 5d - A Comparison of the Deflections, Moments, shears and σ_z in Terms of P for the Line Load Problem.


V_x/P
1
2
3
4
x/h
R
C
$\nu = 1/3$
.25
.50
5c
0.4
0.3
0.2
0.1
$\frac{\sigma_z h}{P}$
1/3
0
2/3
C
1
2
3
4
x/h value on curve
$\nu = 1/3$
5d
z/h
-1/2
-1/4
0
1/4
1/2

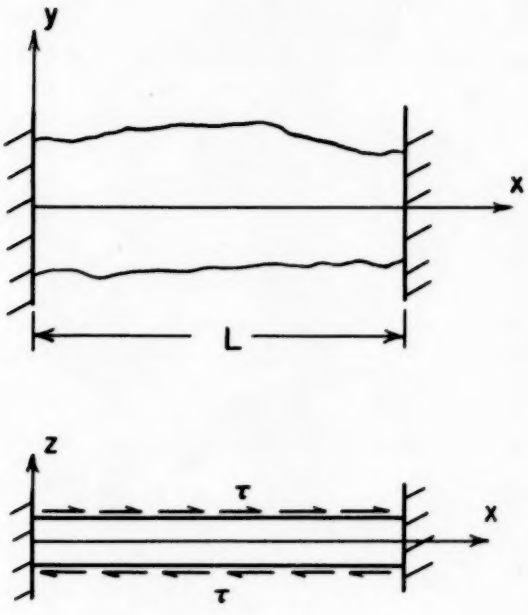
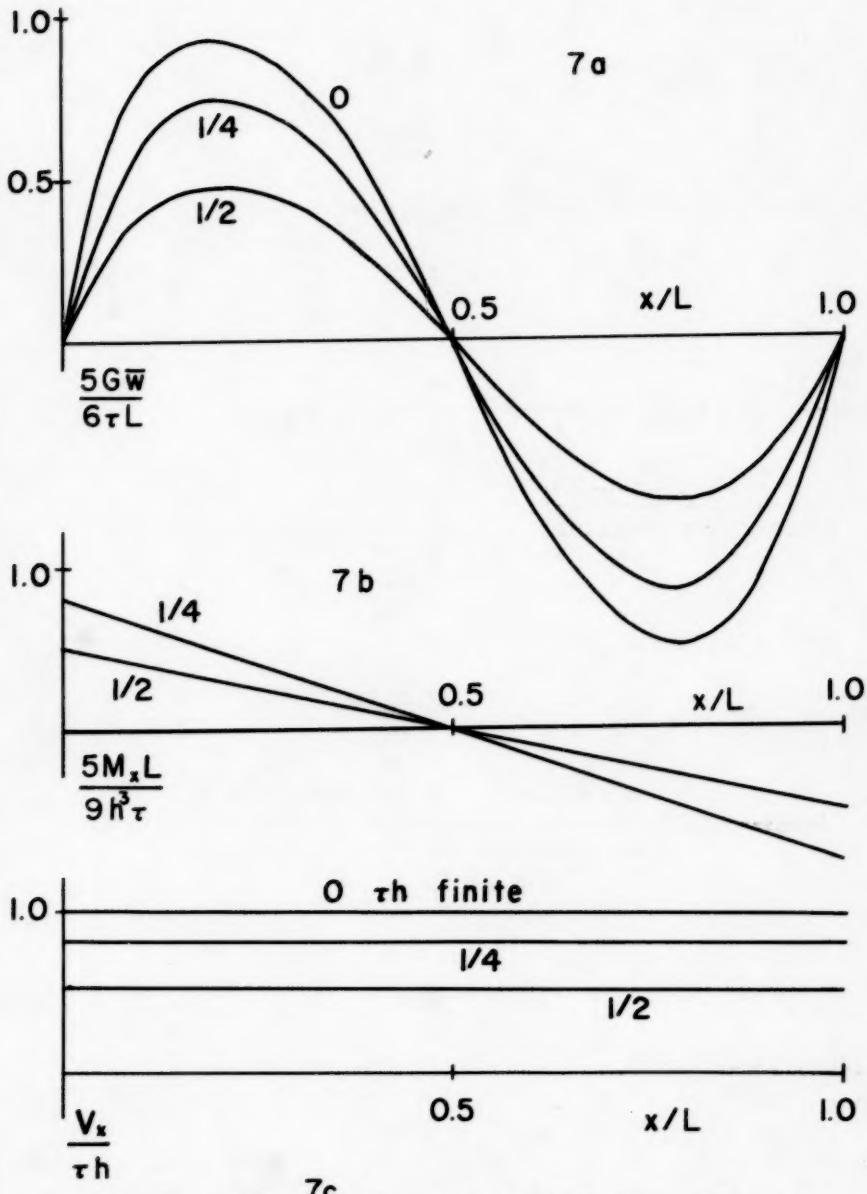


Figure 6 - A Plate with Shear Stresses on the Top and Bottom Faces.



7c

Figures 7a, 7b, 7c - A Comparison of the Deflections, Moments and Shears for the Plate Bent by Shear Stresses. (ν= 1/3; the h/L values are given on the curves)